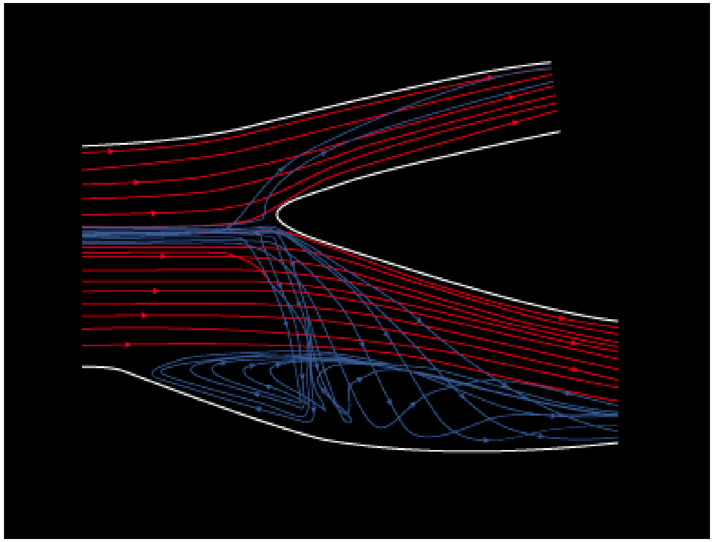
Bernoulli equation



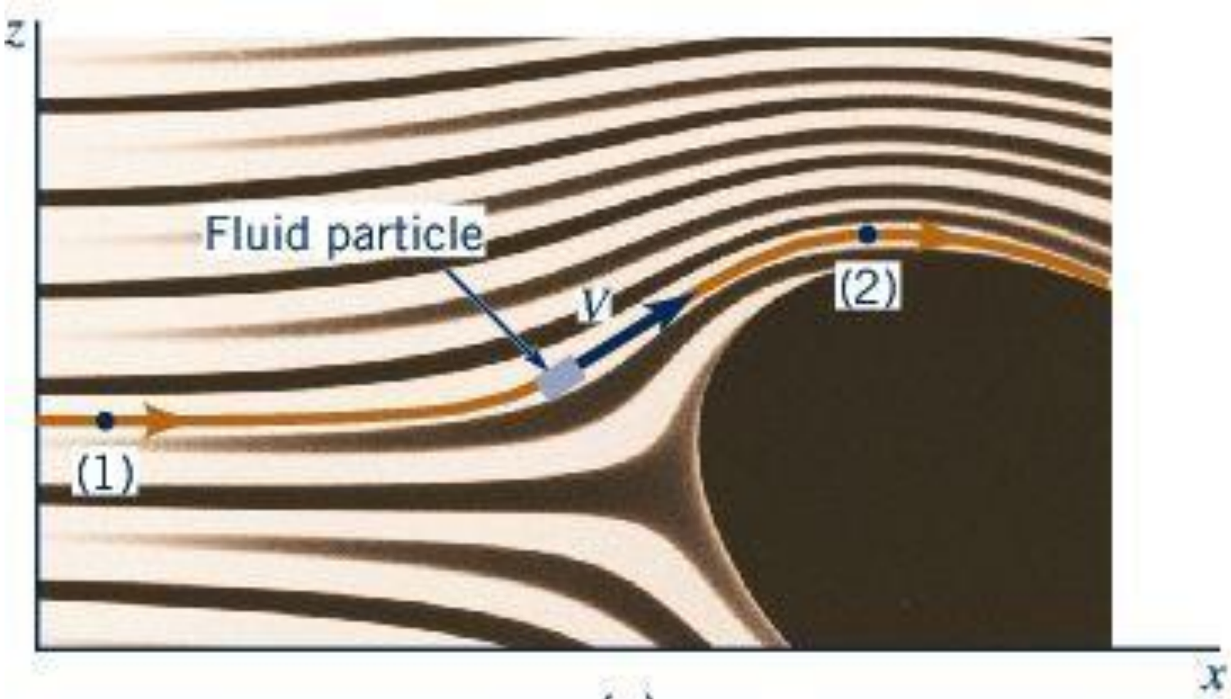
Newton's 2nd law $F = m \cdot a$

Fluid accelerates and decelerates as it moves in the flow field due to the action of:

- **pressure**
- **gravity**
- friction (viscous forces)
- surface tension
- etc.

For an **inviscid fluid (low viscosity or when viscous effects are negligible)**, the main forces to consider are
a) pressure, and
b) gravity

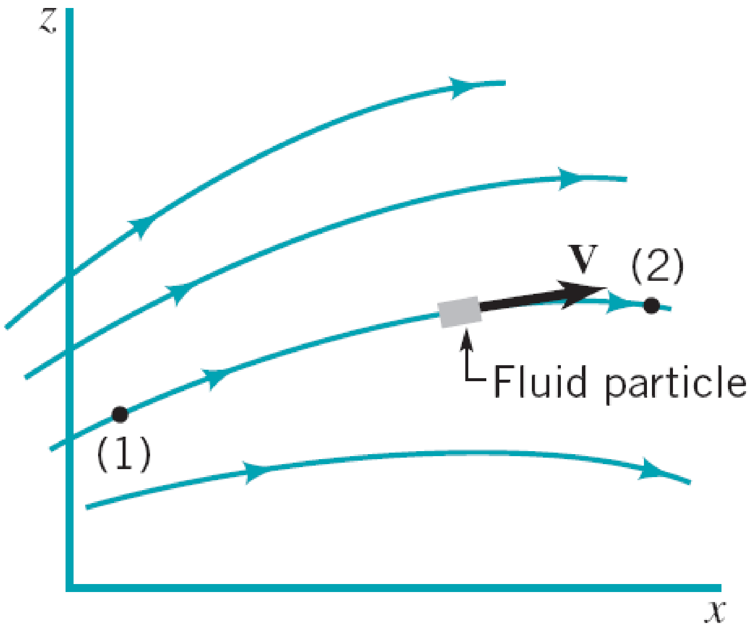
Streamlines in steady flow



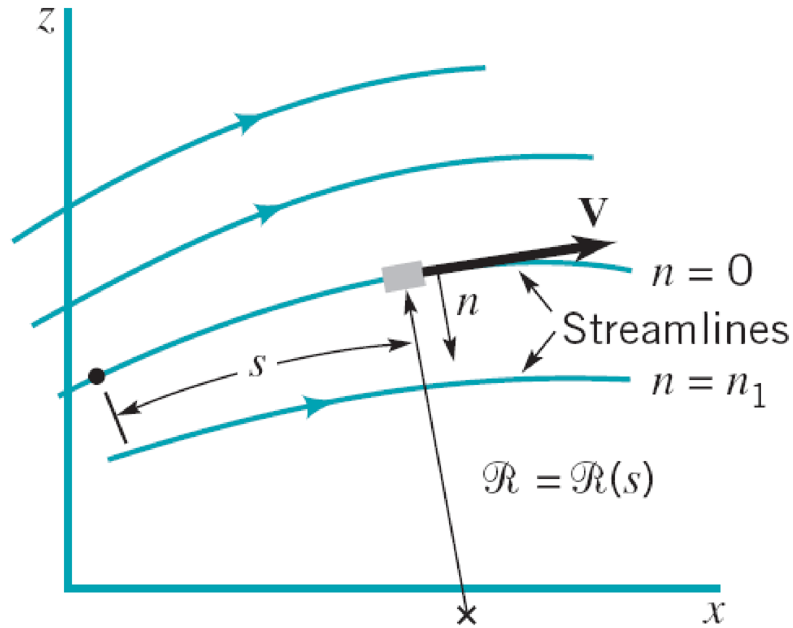
(a)

Acceleration in curvilinear coordinates

Flow in the *x*-*z* plane



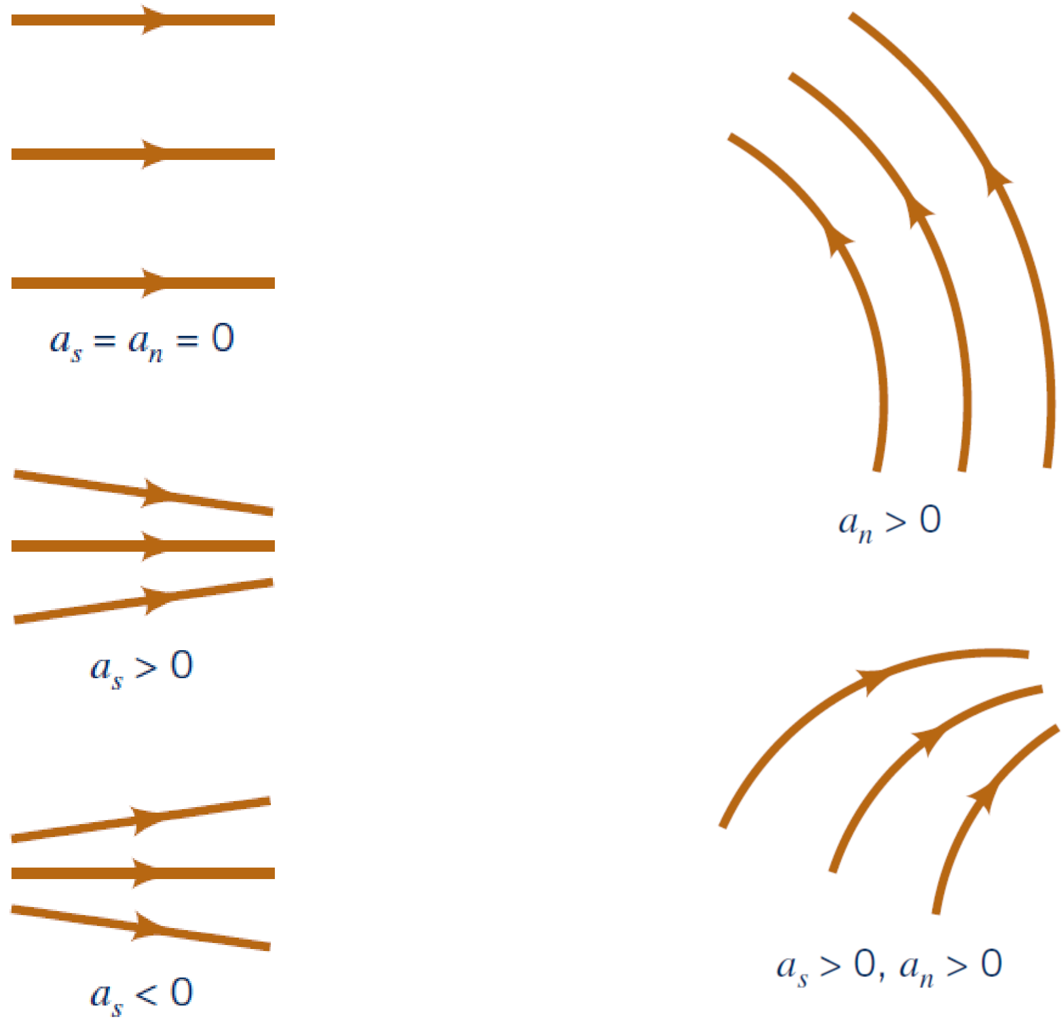
Flow in terms of streamline and normal coordinates



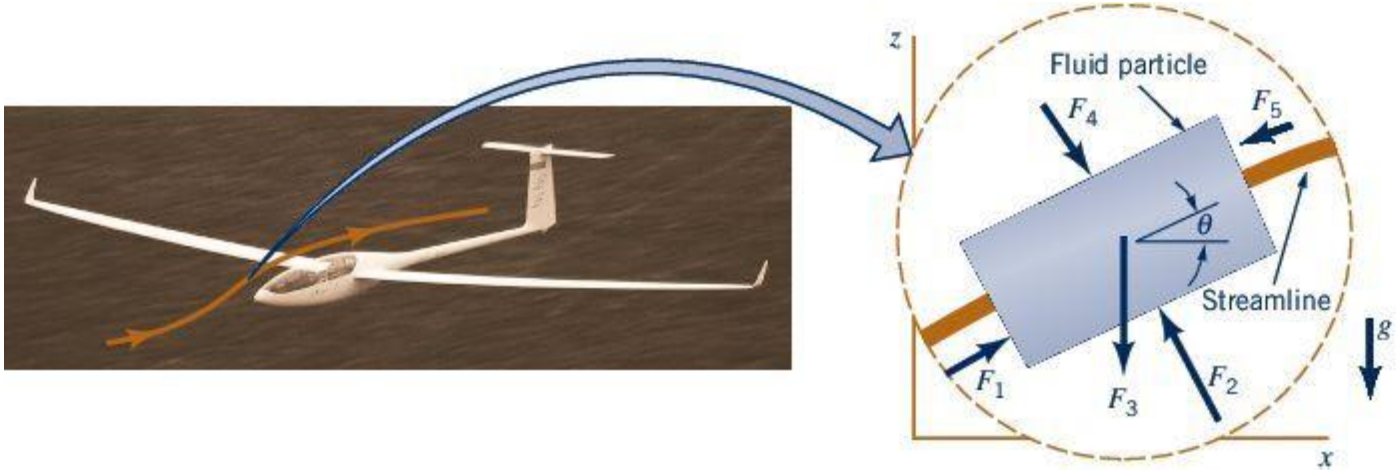
s-component of the acceleration: $a_s = \frac{dV}{dt} = \frac{\partial V}{\partial s} \cdot \underbrace{\frac{\partial s}{\partial t}}_{V} \Rightarrow \underline{\underline{a_s = V \frac{\partial V}{\partial s}}}$

n-component of the acceleration: $\underline{\underline{a_n = \frac{V^2}{\mathcal{R}}}}$

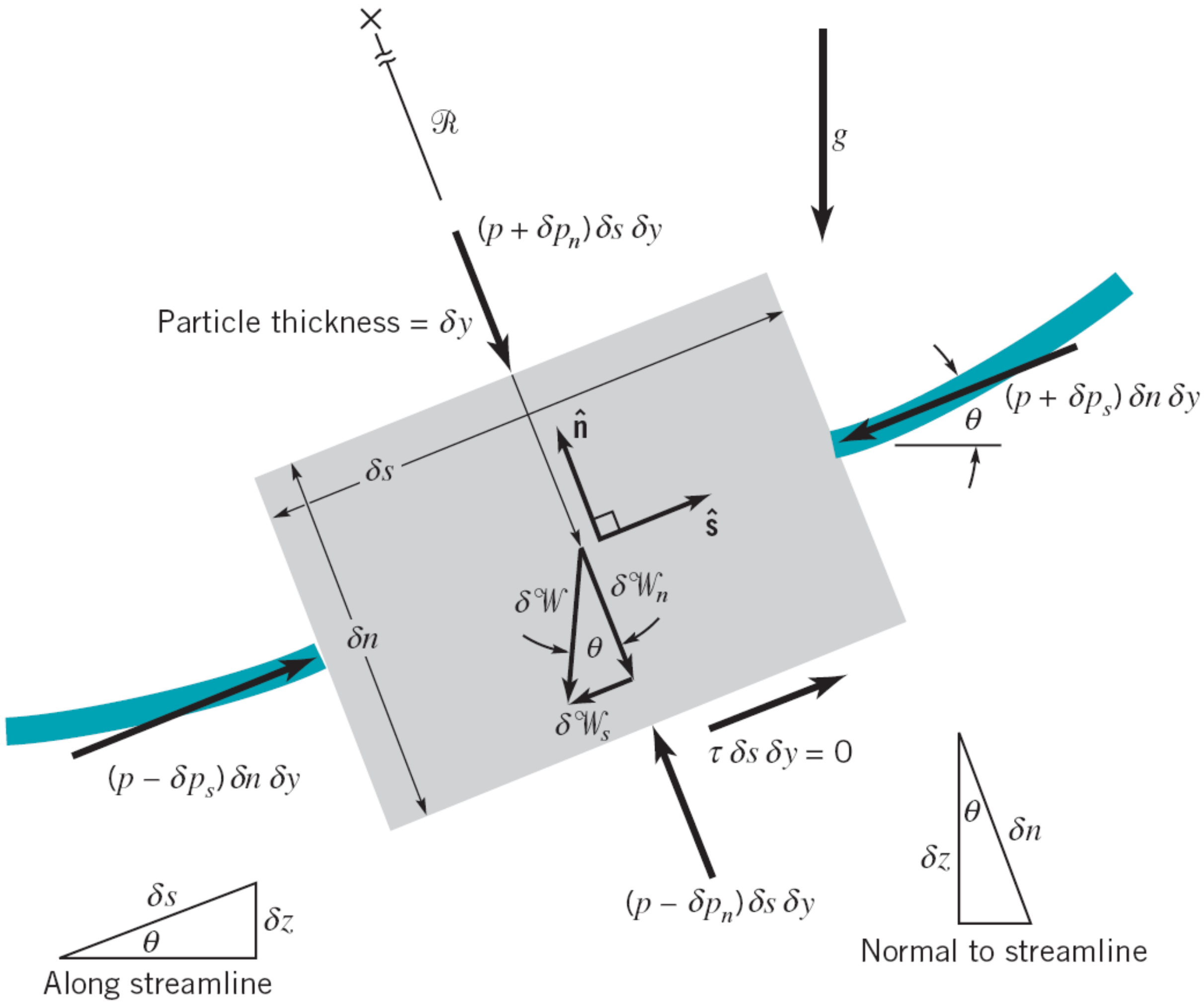
Acceleration in curvilinear coordinates



How to balance gravity and pressure on an isolated fluid particle?

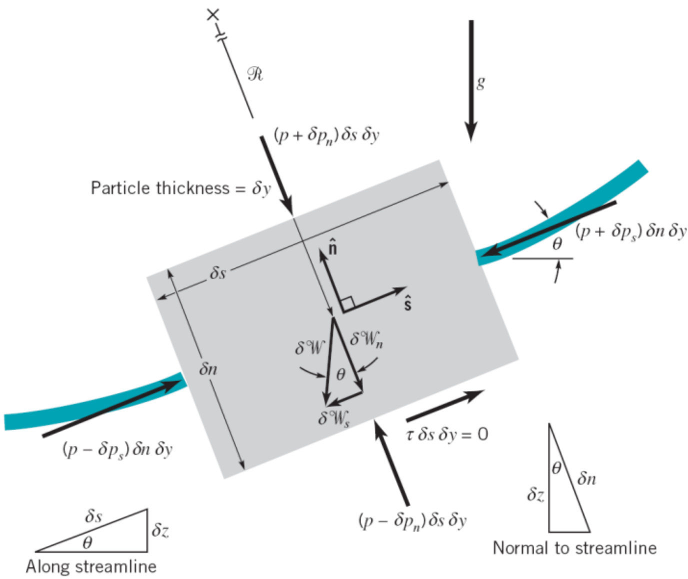


Derivation of Bernoulli's equation



Free-body diagram of a fluid particle for which the important forces are those due to pressure and gravity.

Bernoulli's equation along a streamline



① Assume steady flow $\quad \sum \Delta F_s = \delta m \cdot a_s \quad (1)$

② Assume no friction $\rightarrow$ only pressure & gravity

$$(1) \Rightarrow \left(p - \frac{\partial p}{\partial s}\frac{\delta s}{2}\right)\delta n\,\delta y - \left(p + \frac{\partial p}{\partial s}\frac{\delta s}{2}\right)\delta n\,\delta y - \delta W \sin\theta = \delta m \cdot V\frac{\partial V}{\partial s}$$

$$\Rightarrow -\frac{\partial p}{\partial s}\,\delta s\,\delta n\,\delta y - \gamma\cdot\delta s\,\delta n\,\delta y \cdot \sin\theta = \rho\cdot\delta s\,\delta n\,\delta y\cdot V\frac{\partial V}{\partial s}$$

$$\Rightarrow -\frac{\partial p}{\partial s} - \gamma\cdot\sin\theta = \rho\, V\frac{\partial V}{\partial s}$$

③ Along a streamline: $dV = \frac{\partial V}{\partial s}ds + \frac{\partial V}{\partial n}dn$ ($dn \to 0$, $n = ct$)

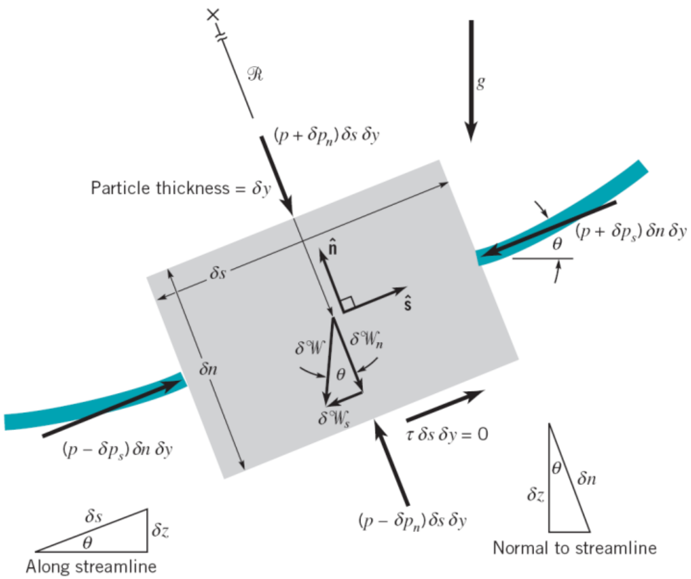
$$\therefore\ -\frac{dp}{ds} - \gamma\sin\theta = \rho\, V\frac{dV}{ds} \Rightarrow -dp - \gamma\underbrace{\sin\theta\, ds}_{dz} = \rho\, V\, dV$$

$$\Rightarrow -dp - \gamma\, dz = \frac{1}{2}\rho\, d(V^2) \Rightarrow dp + \frac{1}{2}\rho\, d(V^2) + \gamma\, dz = 0$$

④ Incompressible fluid: $\rho = ct$ and $\gamma = ct$

$$\therefore\ p + \frac{1}{2}\rho V^2 + \gamma z = \text{constant along a streamline}$$

Bernoulli's equation across a streamline



$$\Sigma \Delta F_n = \delta m \cdot \frac{V^2}{R}$$

$$\Rightarrow \left(p - \frac{\partial p}{\partial n}\frac{\delta n}{2}\right)\delta s\,\delta y - \left(p + \frac{\partial p}{\partial n}\frac{\delta n}{2}\right)\delta s\,\delta y - W\cos\theta = \rho\,\delta s\,\delta n\,\delta y\,\frac{V^2}{R}$$

$$\Rightarrow -\frac{\partial p}{\partial n} - \gamma\cos\theta = \rho\frac{V^2}{R} \quad (1)$$

Note: if gravity is neglected $\frac{\partial p}{\partial n} = -\rho\frac{V^2}{R}$

Across a streamline: $ds = 0 \rightarrow \frac{\partial p}{\partial n} = \frac{dp}{dn}$

$$(1) \Rightarrow -\frac{dp}{dn} - \gamma\cos\theta = \rho\frac{V^2}{R}$$

$$\Rightarrow -dp - \gamma\underbrace{\cos\theta\, dn}_{dz} = \rho\frac{V^2}{R}dn$$

$$\Rightarrow dp + \gamma\, dz + \rho\frac{V^2}{R}dn = 0$$

For incompressible fluid:

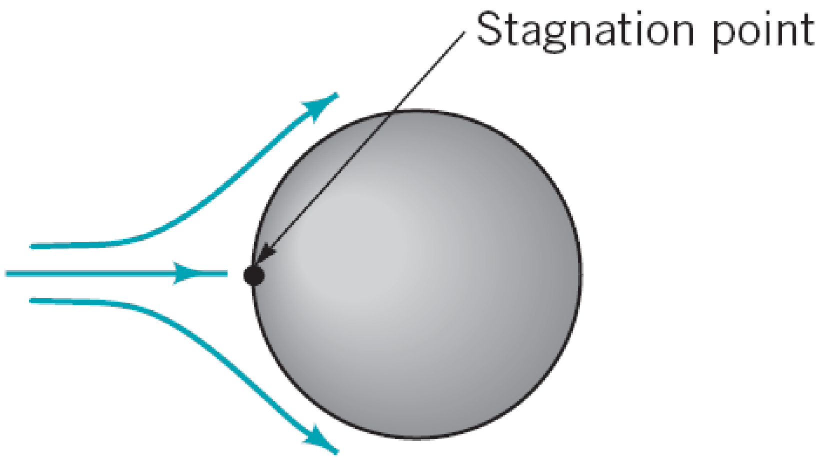
$$p + \gamma z + \rho\int\frac{V^2}{R}dn = ct$$

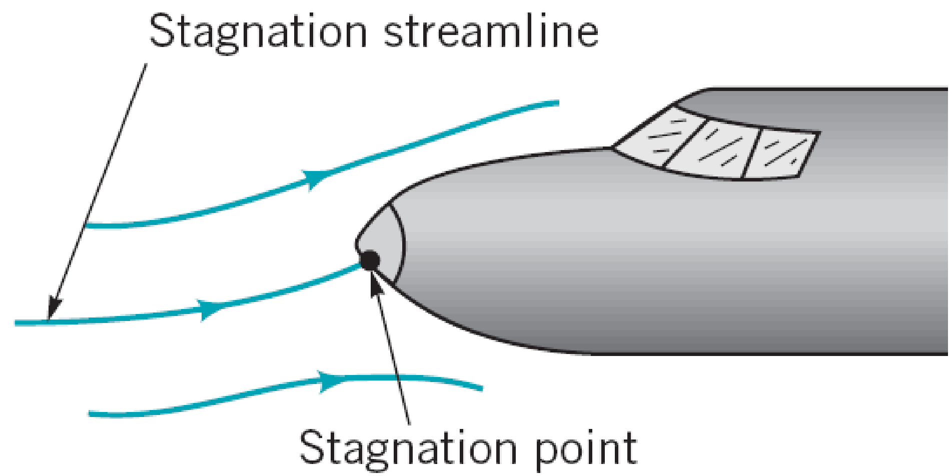
Balancing ball

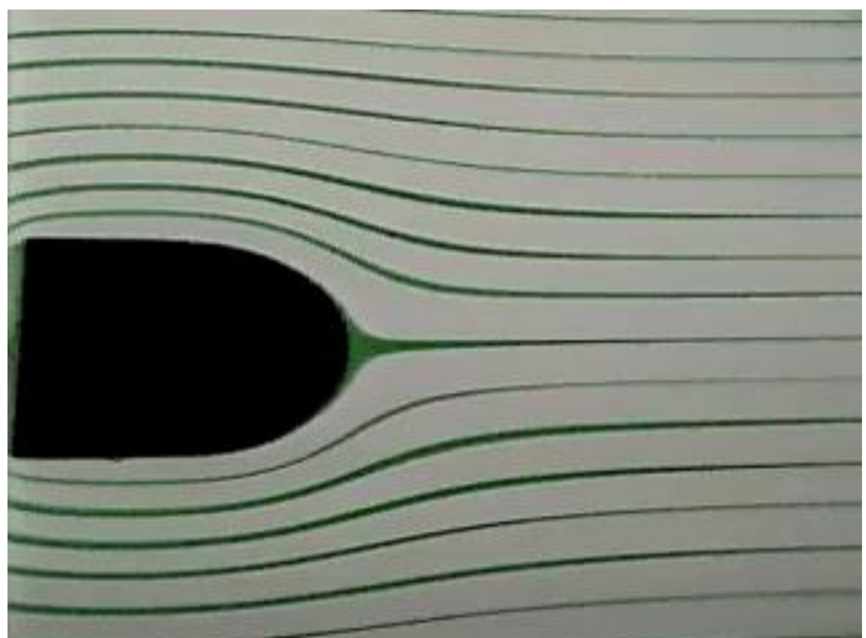
According to the Bernoulli equation, an increase in velocity can cause a decrease in pressure.

The table tennis ball is supported by a jet of air. The net vertical air force is balanced by the ball's weight. If the ball is displaced from the center of the jet, the air velocity past the ball is greater on the side near the jet's center than it is on the side near the jet's edge. Thus, the pressure on the ball is lower near the center, and the ball returns to its stable equilibrium position centered in the air jet

Stagnation points on bodies in flowing fluids

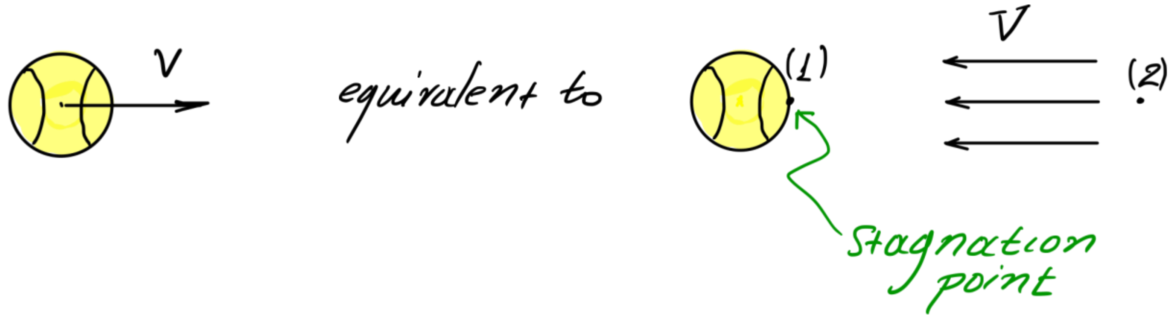






Example: pressure on a tennis ball

Andy Roddick serves at 250 km/hr.
What is the pressure on the tip of the ball?



Bernoulli between points (1) and (2):

$$P_1 + \frac{1}{2}\rho V_1^2 + \gamma z_1 = P_2 + \frac{1}{2}\rho V_2^2 + \gamma z_2$$

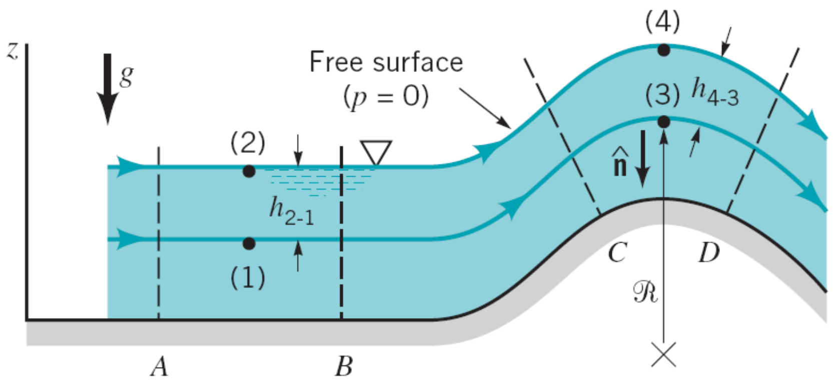
$z_1 = z_2$; $V_1 = 0$ (stagnation point; $P_2 = P_{atm} = 0$

$$\therefore P_1 = \frac{1}{2}\rho V_2^2 = \frac{1}{2} \cdot 1.23\ \mathrm{kg/m^3} \times \left(250\ \mathrm{km/hr} \cdot \frac{1000\ \mathrm{m/km}}{3600\ \mathrm{s/hr}}\right)^2$$

$$\Rightarrow \underline{\underline{P_1 = 2.97\ \mathrm{kPa}}}$$

Compared to $P_{atm} \simeq 101$ kPa, this is only a slight (< 3%) pressure increase.

Example: Bernoulli equation across a streamline



Determine the pressure variation between:

a) points (1) and (2)

b) points (3) and (4)

Bernoulli eq. across streamlines:

$$dp + \gamma dz + \rho \frac{V^2}{R} dn = 0 \qquad (1)$$

For portion AB: $R = \infty$, hence

$$dp + \gamma dz = 0 \Rightarrow \int_{P_1}^{P_2} dp + \int_{z_1}^{z_2} \gamma dz = 0$$

$$\Rightarrow P_2 - P_1 + \gamma (z_2 - z_1) = 0 \qquad (P_2 = 0)$$

$$\Rightarrow P_1 = \gamma (z_2 - z_1) = \gamma h_{2-1} \qquad \text{hydrostatic}$$

For portion CD: $dn = -dz$ on the vertical line

Integrating eq (1) from (3) to (4):

$$\int_{P_3}^{P_4} dp + \gamma \int_{z_3}^{z_4} dz + \rho \int_{z_3}^{z_4} \frac{V^2}{R} (-dz) = 0$$

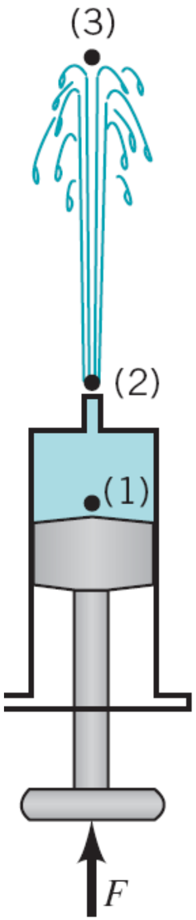
$$\Rightarrow P_4 - P_3 + \gamma \underbrace{(z_4 - z_3)}_{h_{4-3}} - \rho \int_{z_3}^{z_4} \frac{V^2}{R} dz = 0 \qquad (P_4 \to 0)$$

$$\Rightarrow P_3 = \gamma h_{4-3} - \rho \int_{z_3}^{z_4} \frac{V^2}{R} dz$$

Question:

P_3 smaller than hydrostatic. Why?

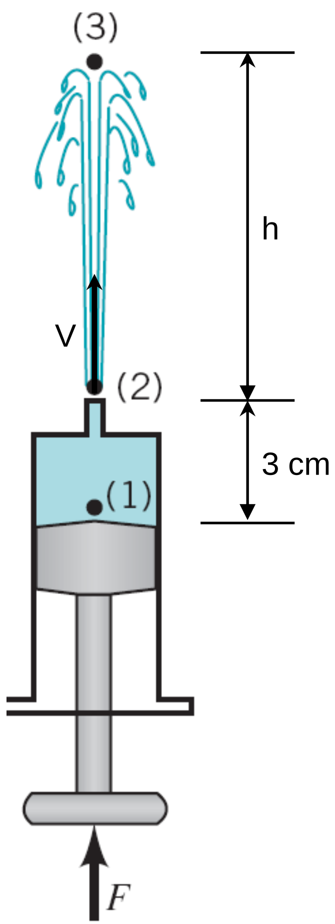
Kinetic, potential and pressure energy



	Energy Type		
Point	Kinetic $\rho V^2/2$	Potential γz	Pressure p
1	Small	Zero	Large
2	Large	Small	Zero
3	Zero	Large	Zero

Kinetic, potential and pressure energy

V = 2 m/s
P_1 = ?
h = ?



Bernoulli between points (2) ∧ (3)

$$\cancel{P_2} + \frac{1}{2}\rho V_2^2 + \gamma z_2 = \cancel{P_3} + \frac{1}{2}\rho \cancel{V_3^2} + \gamma z_3$$

($P_2 \to 0$, $P_3 \to 0$, $V_3 \to 0$)

$$\Rightarrow \frac{1}{2}\rho V_2^2 = \gamma (z_3 - z_2) = \gamma h = \rho g h$$

$$\Rightarrow h = \frac{V_2^2}{2g} = \frac{(2\ m/s)^2}{2 \times 9.8\ m/s^2} = \underline{\underline{0.205\ m}}$$

Bernoulli between points (1) ∧ (3)

$$P_1 + \frac{1}{2}\rho \cancel{V_1^2} + \gamma z_1 = \cancel{P_3} + \frac{1}{2}\rho \cancel{V_3^2} + \gamma z_3$$

($V_1 \approx 0$ quasi-static, $P_3 \to 0$, $V_3 \to 0$)

$$\Rightarrow P_1 = \gamma (z_3 - z_1) = 9.8\ kN/m^3 \cdot (0.205 + 0.03)\ m$$

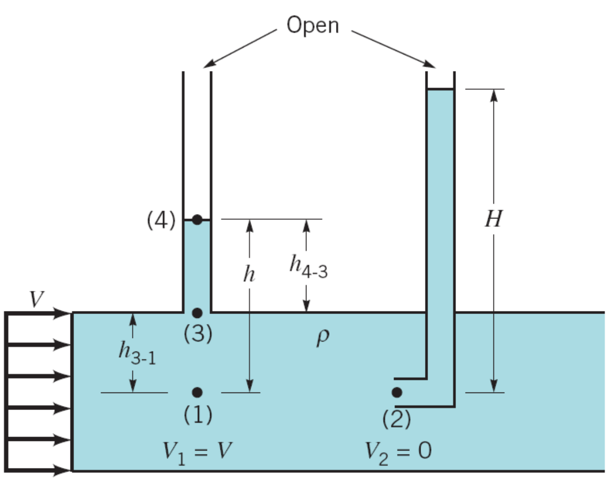
$$\Rightarrow \underline{\underline{P_1 = 2.30\ kPa}}$$

Static, stagnation, total and dynamic pressure

$$\underbrace{P}_{\text{work by pressure forces}} + \underbrace{\tfrac{1}{2}\rho V^2}_{\text{kinetic energy}} + \underbrace{\gamma z}_{\text{potential energy}} = ct$$

One type of energy can be transformed to another

$$\underbrace{P}_{\text{static pressure}} + \underbrace{\tfrac{1}{2}\rho V^2}_{\text{dynamic pressure}} + \underbrace{\gamma z}_{\text{hydrostatic pressure}} = ct$$



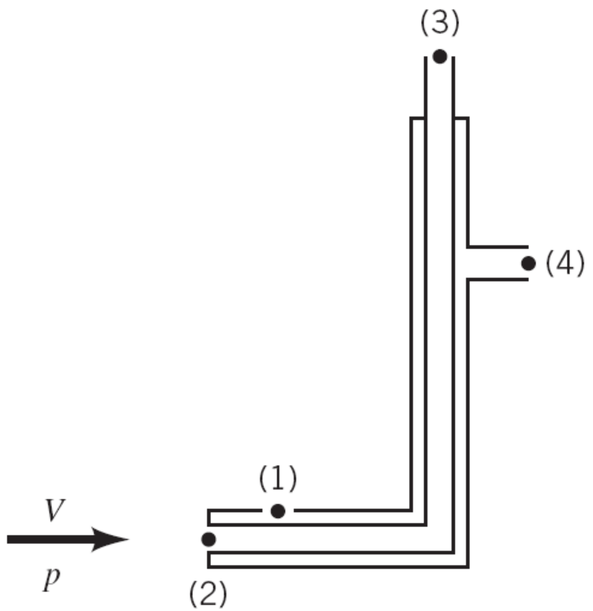
$$P_1 + \tfrac{1}{2}\rho V_1^2 + \cancel{\gamma z_1} = P_2 + \tfrac{1}{2}\rho \cancel{V_2^2} + \cancel{\gamma z_2}$$

($V_2 \to 0$)

$$\Rightarrow \quad P_2 = P_1 + \underbrace{\tfrac{1}{2}\rho V_1^2}_{\text{dynamic pressure}}$$

- P_2: stagnation pressure
- P_1: static pressure
- $\tfrac{1}{2}\rho V_1^2$: dynamic pressure

The Pitot-static tube.



P_1 = static pressure

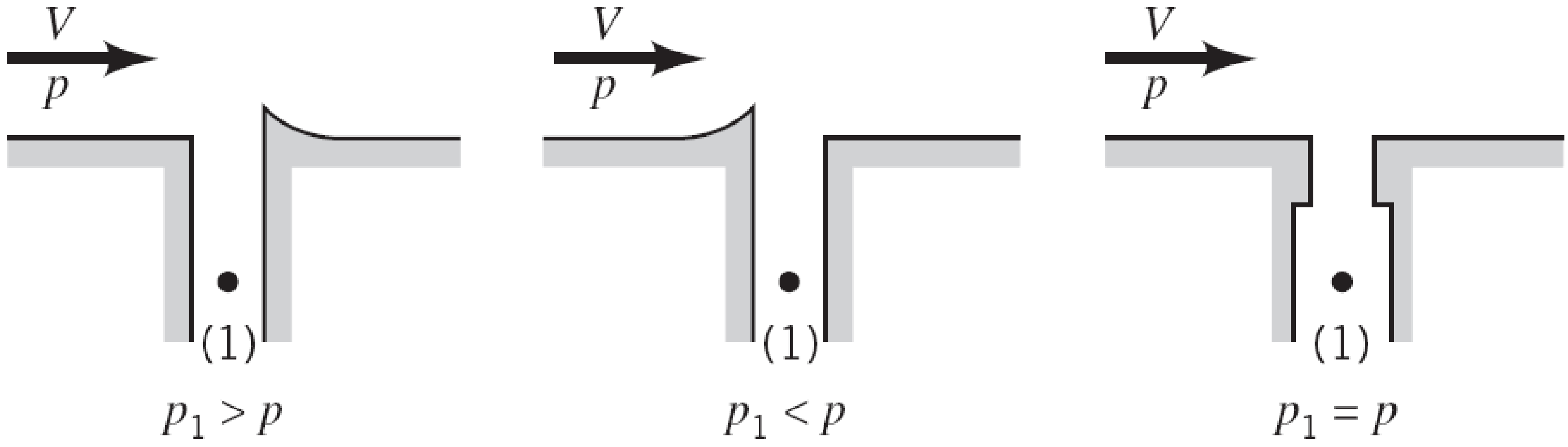
P_2 = stagnation pressure

If elevation changes are small,

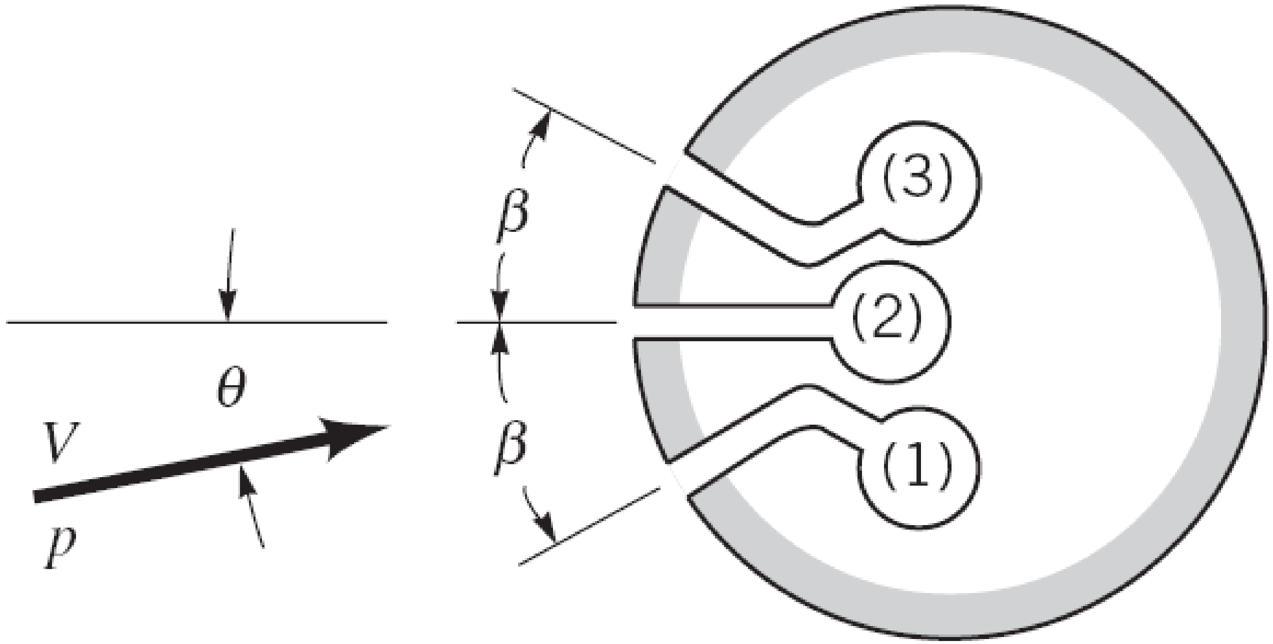
$$P_2 = P_1 + \frac{1}{2}\rho V^2$$

$$\Rightarrow \quad V = \sqrt{\frac{2(P_2 - P_1)}{\rho}}$$

Incorrect and correct design of static pressure taps.



Cross-section of a directional-finding Pitot-static tube.

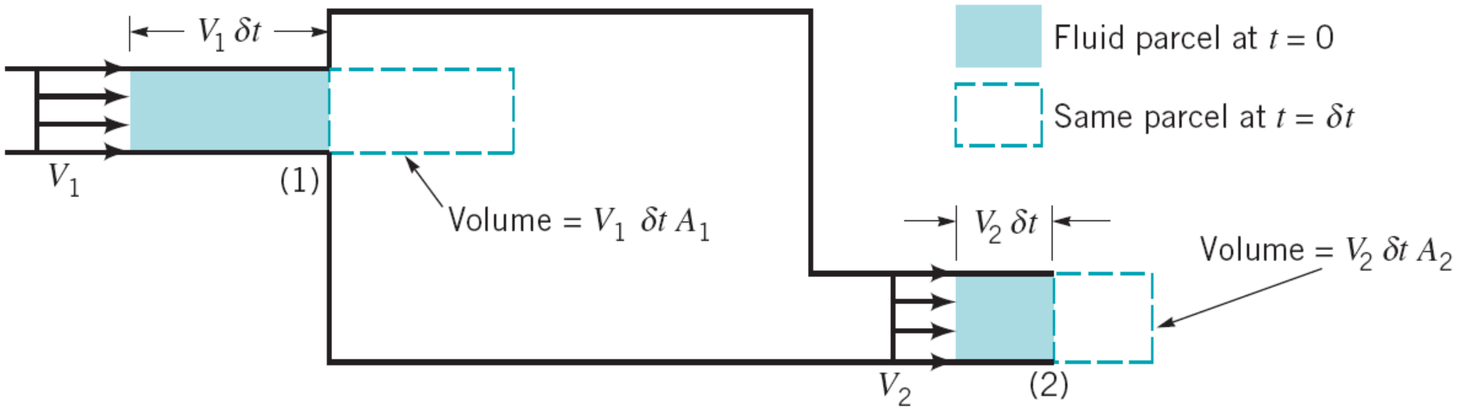


If $\theta = 0$

$$p_1 = p_3 = p$$

$$p_2 = p + \frac{1}{2}\rho V^2$$

Continuity principle



Volume flow rate in: $Q_1 = \dfrac{V_1 A_1 \delta t}{\delta t} = V_1 A_1$

Volume flow rate out: $Q_2 = \dfrac{V_2 A_2 \delta t}{\delta t} = V_2 A_2$

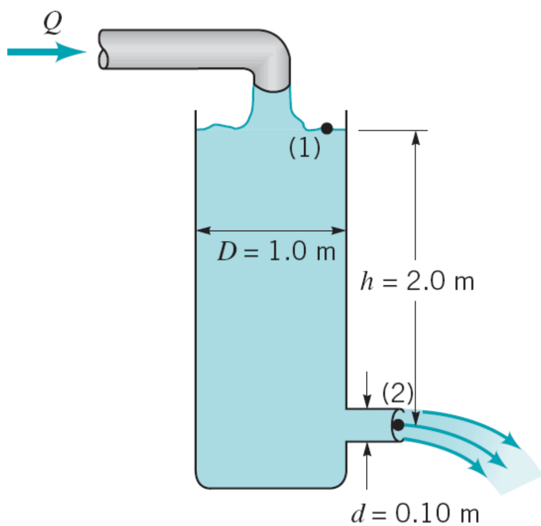
No accumulation: mass in = mass out

$$\rho_1 V_1 A_1 = \rho_2 V_2 A_2 = \dot{m} = ct$$

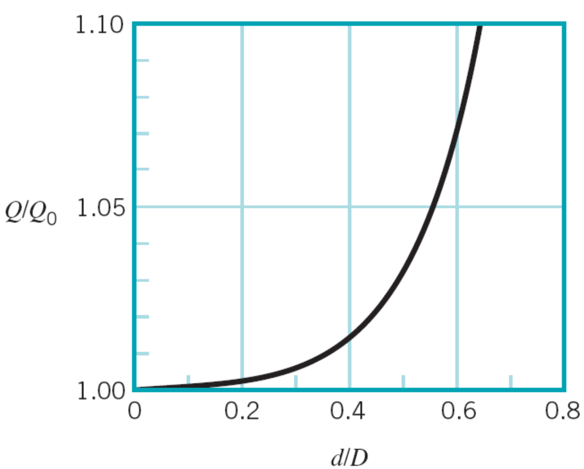
When the fluid is <u>incompressible</u>, $\rho_1 = \rho_2 = ct$

$$\therefore \quad Q_1 = V_1 A_1 = V_2 A_2 = Q_2$$

Example



(a)



(b)

$$p_1 + \frac{1}{2}\rho V_1^2 + \gamma z_1 = p_2 + \frac{1}{2}\rho V_2^2 + \gamma z_2$$

($p_1 \to 0$, $z_1 = h$, $p_2 \to 0$, $z_2 \to 0$)

$$\frac{1}{2}\rho V_1^2 + \gamma h = \frac{1}{2}\rho V_2^2 \qquad (1)$$

Continuity: $A_1 \cdot V_1 = A_2 V_2$

$$\Rightarrow V_1 = \frac{A_2}{A_1} V_2 = \left(\frac{d}{D}\right)^2 V_2 \qquad (2)$$

$$(1) \wedge (2) \Rightarrow \frac{1}{2}\rho \left(\frac{d}{D}\right)^4 V_2^2 + \rho g h = \frac{1}{2}\rho V_2^2$$

$$\Rightarrow V_2 = \sqrt{\frac{2gh}{1-\left(\frac{d}{D}\right)^4}} = \sqrt{\frac{2 \times 9.81 \times 2}{1-\left(\frac{0.1}{1}\right)^4}}\ \mathrm{m/s}$$

$$\Rightarrow V_2 = 6.26\ \mathrm{m/s}$$

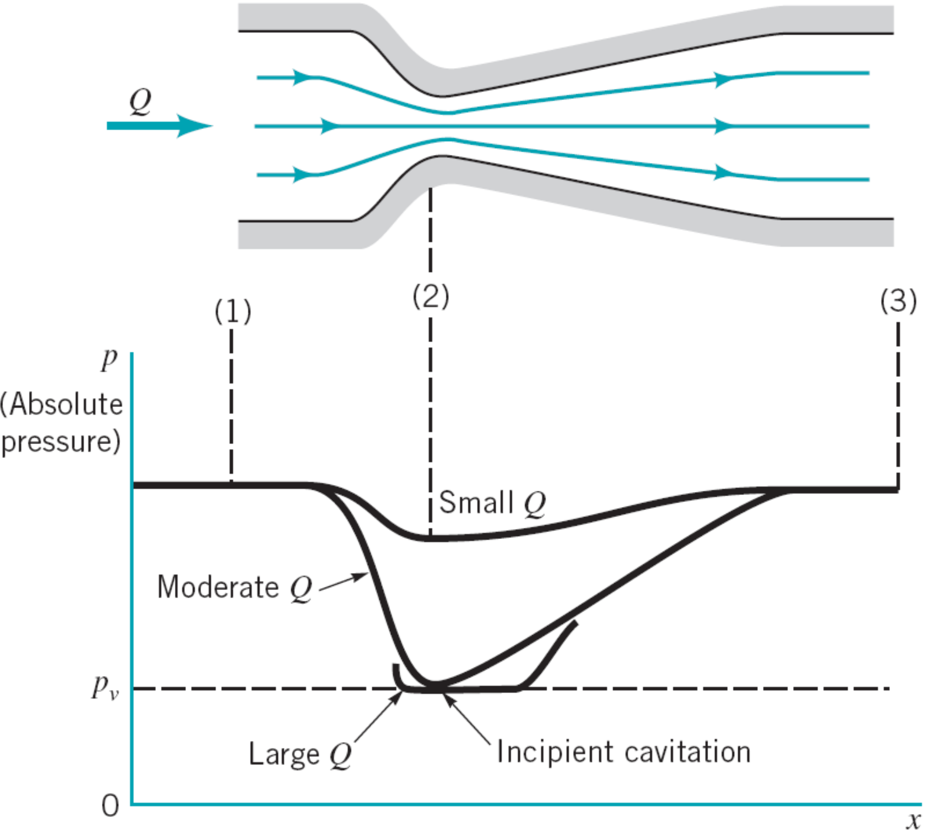
$$\therefore Q = A_1 V_1 = A_2 V_2 = \frac{\pi}{4}(0.1\,\mathrm{m})^2 \cdot (6.26\ \mathrm{m/s}) = 0.0492\ \mathrm{m^3/s}$$

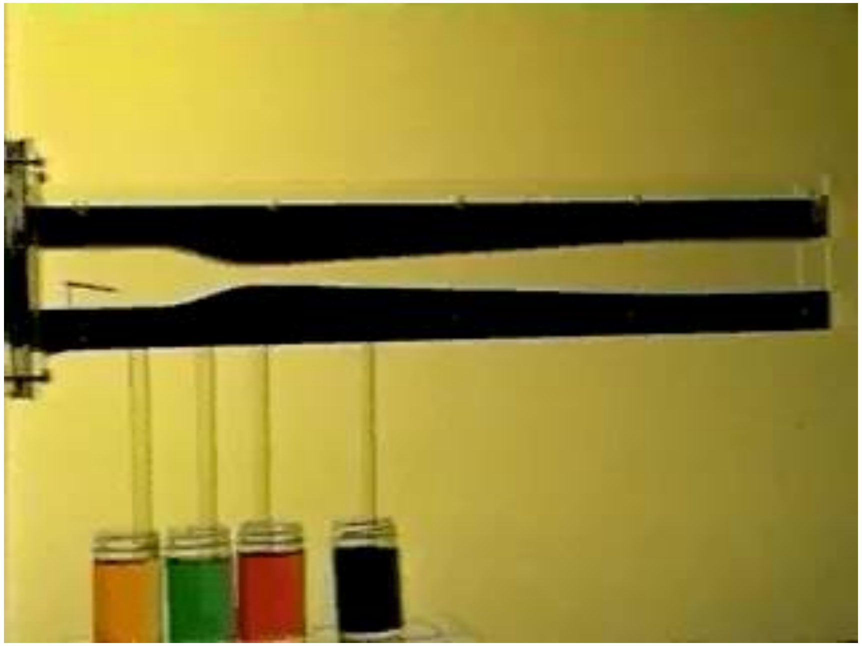
Remark: if $d \ll D \Rightarrow A_2 \ll A_1 \Rightarrow V_1 \ll V_2$

then $V_2 = \sqrt{2gh}$ and

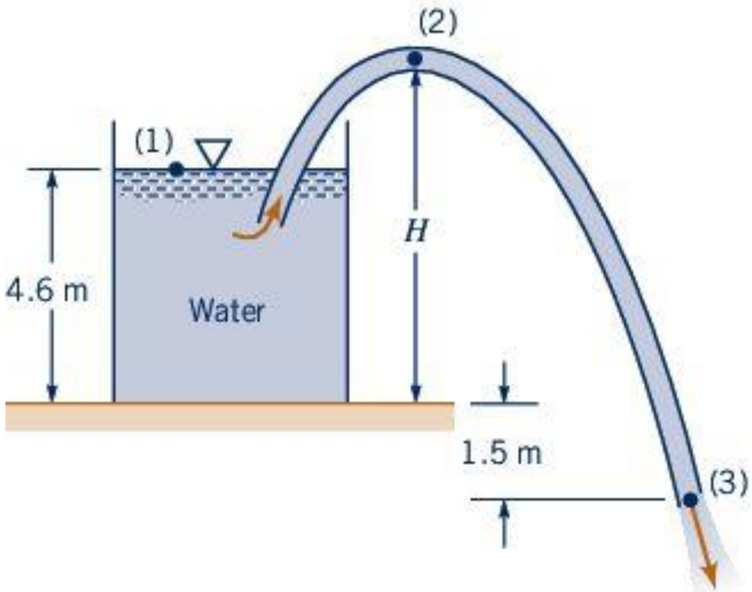
$$\frac{Q}{Q_0} = \frac{V_2}{V_2|_{d \ll D}} = \frac{1}{\sqrt{1-\left(\frac{d}{D}\right)^4}}$$

Pressure variation and cavitation in a variable area pipe





Example



1) Bernouilli 1 - 3

$$\cancel{p_1} + \frac{1}{2}\rho \cancel{V_1^2} + \gamma z_1 = \cancel{p_3} + \frac{1}{2}\rho V_3^2 + \gamma z_3$$

$$V_3 = \sqrt{2g(z_1 - z_3)} = V_2$$

Continuity:
A_2=A_3 → V_2=V_3

$$V_2 = \sqrt{2\left(9.81\frac{m}{s^2}\left(4.6-(-1.5)\right)m\right.} = 10.9\frac{m}{s}$$

2) Bernouilli 1 - 2

$$\cancel{p_1} + \frac{1}{2}\cancel{\rho V_1^2} + \gamma z_1 = p_2 + \frac{1}{2}\rho V_2^2 + \gamma z_2$$

$$p_2 = \gamma(z_1 - z_2) - \frac{1}{2}\rho V_2^2$$

Until what height H can water be siphoned out of the tank before cavitation occurs?

3) Cavitation when lowest pressure reaches p=p_v

p_v = 1.765 kPa for water at 15^0 C

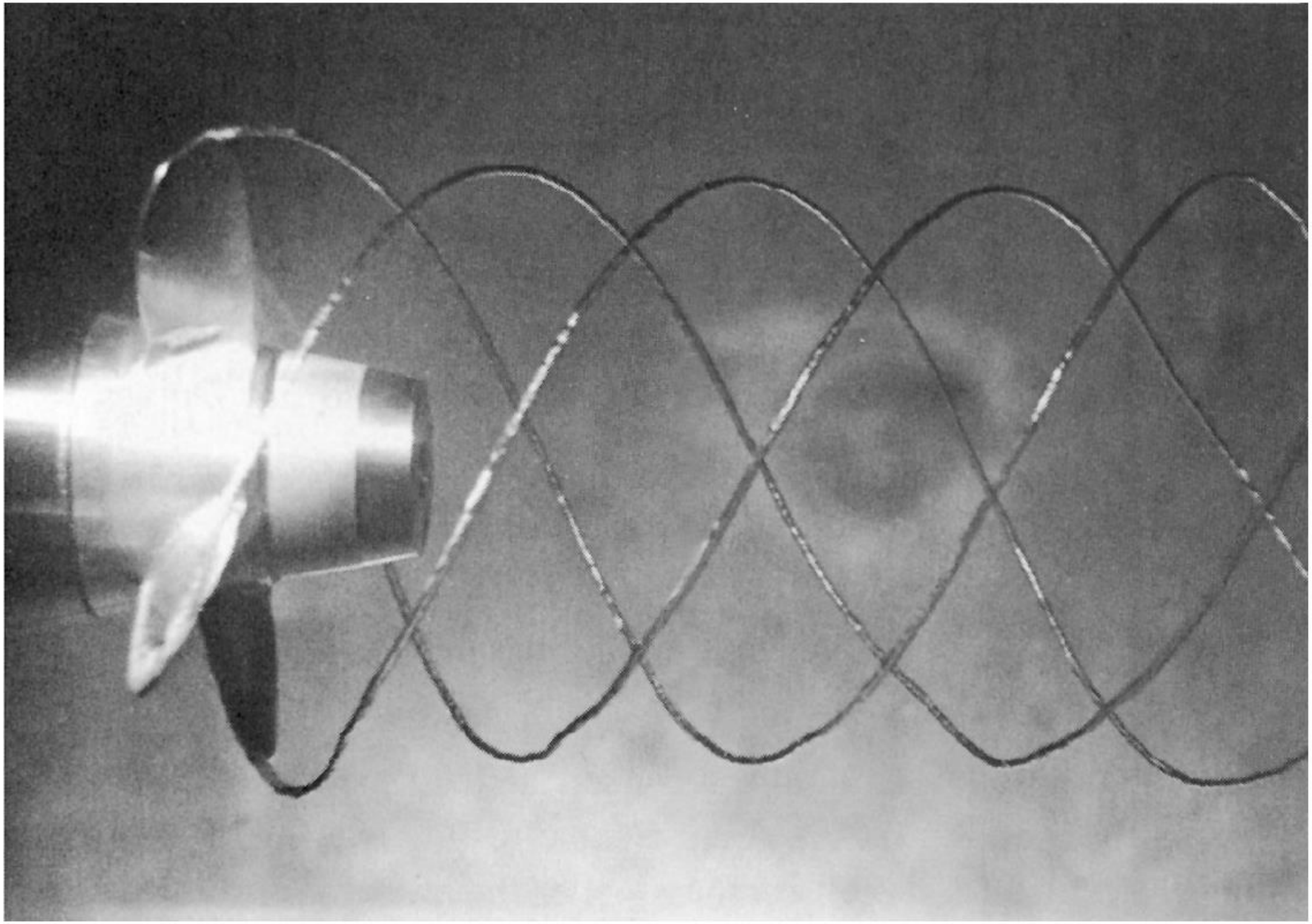
p_2 = 1.765 kPa – 101.3 kPa

P1=0 →
gage pressure

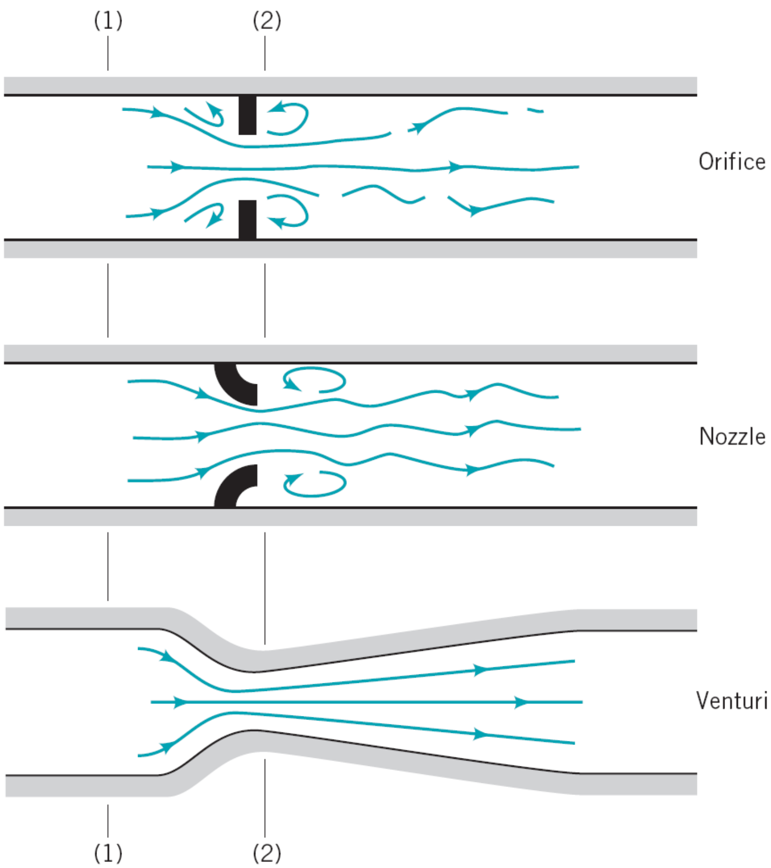
1) + 2) + 3): -99.5 *kPa* = 9.8 *kN/m^3* (4.6 *m* - H) – ½(1000 *kg/m^3*)(10.9 *m/s*)^2

→ H = 8.6 m

Tip cavitation from a propeller



Flowrate measurement



$$P_1 + \frac{1}{2}\rho V_1^2 + \gamma z_1 = P_2 + \frac{1}{2}\rho V_2^2 + \gamma z_2 \quad (1)$$

Continuity: $Q = A_1 V_1 = A_2 V_2$ (2)

$$(1) \wedge (2) \Rightarrow \quad P_1 + \frac{1}{2}\rho\left(\frac{Q}{A_1}\right)^2 = P_2 + \frac{1}{2}\left(\frac{Q}{A_2}\right)^2$$

$$\Rightarrow \quad P_1 - P_2 = \frac{1}{2}\rho Q^2\left(\frac{1}{A_2^2} - \frac{1}{A_1^2}\right)$$

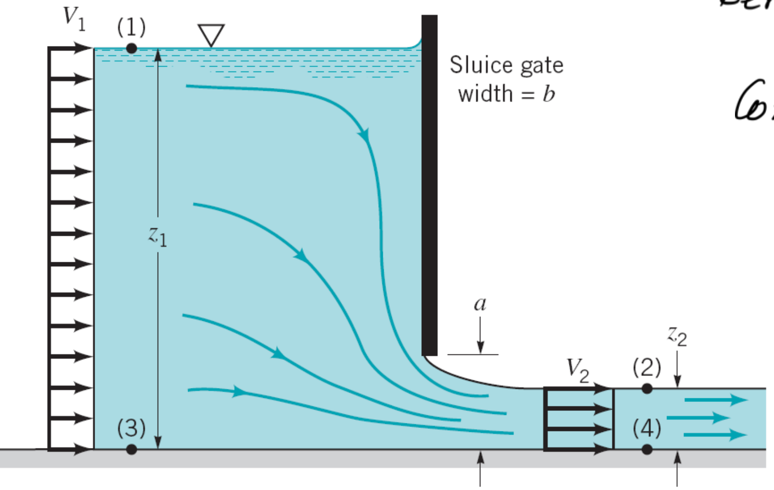
$$\Rightarrow \quad Q^2 = \frac{2(P_1 - P_2)}{\rho \frac{1}{A_2^2}\left(1 - \frac{A_2^2}{A_1^2}\right)}$$

$$\Rightarrow \quad Q = A_2\sqrt{\frac{2(P_1 - P_2)}{\rho\left(1 - \frac{A_2^2}{A_1^2}\right)}}$$

Remark: $Q_{actual} = C_f \cdot Q$

where $C_f < 1$ $(0.6 < C_f < 0.98)$

Sluice gate



Bernoulli: $\cancel{P_1}^{\to 0} + \frac{1}{2}\rho V_1^2 + \gamma z_1 = \cancel{P_2}^{\to 0} + \frac{1}{2}\rho V_2^2 + \gamma z_2$

Continuity: $Q = (b z_1) V_1 = (b z_2) V_2$

$$\Rightarrow \frac{1}{2}\rho\left(\frac{Q}{b z_1}\right)^2 + \gamma z_1 = \frac{1}{2}\rho\left(\frac{Q}{b z_2}\right)^2 + \gamma z_2$$

$$\Rightarrow \frac{1}{2}\frac{1}{(b z_2)^2}\cdot\left[1 - \left(\frac{z_2}{z_1}\right)^2\right] Q^2 = g(z_1 - z_2)$$

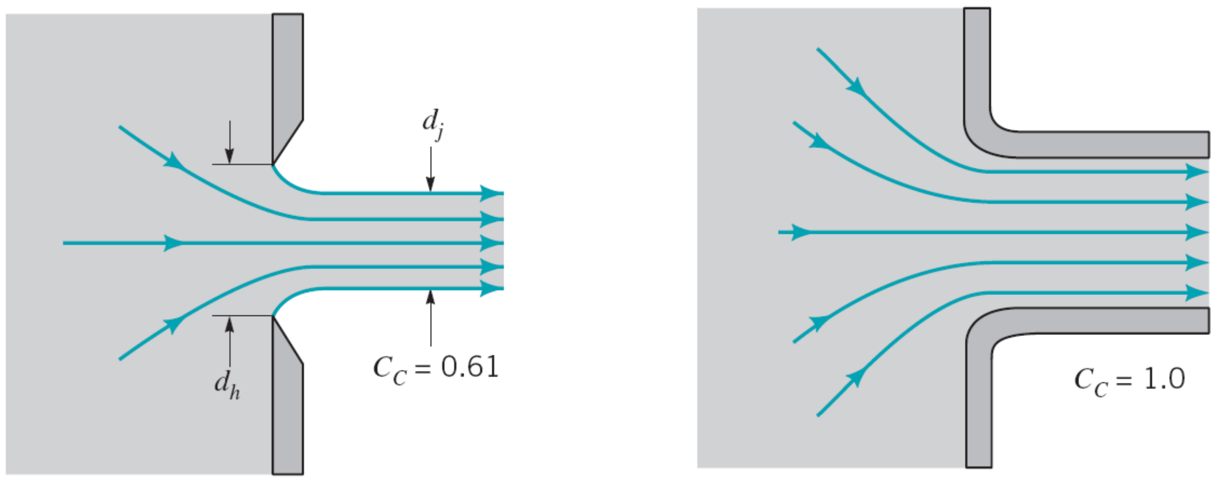
$$\Rightarrow Q = b z_2 \cdot \sqrt{\frac{2g(z_1 - z_2)}{1 - \left(\frac{z_2}{z_1}\right)^2}}$$

If $z_2 \ll z_1 \Rightarrow$ $Q \simeq b z_2 \sqrt{2g z_1}$

Contraction coefficient: $C_c = \frac{z_2}{a}$ $\therefore$ $Q \simeq C_c\, b a \sqrt{2g z_1}$

If $a/z_1 < 0.2$, $C_c \simeq 0.61$

Typical contraction coefficients for different outlet configurations



$$C_C = A_j/A_h = (d_j/d_h)^2$$